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REPLY TO "CRITICISMS".

BY PROF. DE VOLSON WOOD, HOBOKEN, N. J.

IT is fortunate that writers differ in their manner of presenting a subject. Some amplify to diffuseness, others condense to abruptness, while a few hit the happy mean. I am pleased to see Sec. 9 of pp. 36 and 127, of the preceding volume of the ANALYST, in Mr. Christie's own clear and adequately emphatic style; and although that topic might be further amplified, it is deemed unnecessary to do so for present purposes.

In regard to the definition given to $U\beta$, we might have taken refuge under the plea that it was a new meaning applied to U; but as I considered it in conflict with that given by Hamilton,* I at once abandoned it. Were it simply *worthless*, it would be *harmless*. The critic's remark in regard to the term *implies*, is attacking a mode of expression rather than the subject-matter under consideration.

On p. 67 we wrote

$$k=\frac{j}{i}, \qquad -k=\frac{i}{j};$$

from which it was inferred that "the reciprocal of the fraction *changes the sign* of the vector axis instead of producing its reciprocal." Although this was criticised, we observe that *no other conclusion can be drawn from the principles stated up to that point.* Positive rotation only had been considered. Hamilton says (*Elements*, p. 121) "It is evident that the angle of the reciprocal remains *unchanged*, but that the axis is *reversed* in direction", which agrees exactly with our statement. But afterwards—pp. 68 and 69 of our article—the notation for and effect of negative rotation was considered, and it was shown that $k^{-1} = -k$ and, following Hamilton's notation, we write $k^{-1}=\frac{1}{k}$, and *call* the expression 'the reciprocal of a vector'. It was shown that k^{-1} implies (or *expresses*) a negative quadrantal rotation about a positive axis; so that we have

$$k=\frac{j}{i}, \qquad -k=k^{-1}=\frac{i}{j};$$

hence, as stated at p. 128, "When the direction of rotation of the fraction and its reciprocal are in opposite senses, the reciprocal of the axis is the axis of the reciprocal." *This is correct.* The remark of the critic that "It is a matter of easy and direct perception" makes it none the less correct.

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*Prior to writing my articles, my efforts at getting even a sight of Hamilton's *Elements* had been unsuccessful, but later I had access to a copy. It is clear, concise, comprehensive.

Having failed to draw this inference after negative rotation had been considered—which fact was called to mind by reading Mr. Christie's communication—the "emendation" referred to was made expressly for the purpose of calling the attention of the reader, by the contrast, to the fact that negative rotation had not previously been considered. Tait says "The versor of the reciprocal differs from the versor of the quaternion merely by the reversal of its representative angle" (Tait's *Quaternions*, p. 28), which agrees with the conclusion above. Hamilton calls these *Conjugate versors* (*Lect's*, p. 87). As a *result*, it is true that the axis of the reciprocal is the reciprocal of the axis, but in the operation of producing the reciprocal *one* or *the other* of the above processes will necessarily be used. (Lectures, p. 123.)

In regard to the expression

$$kk = \frac{-j}{i} \cdot \frac{i}{j} = \frac{-i}{i} = -1,$$

the critic states that it should be written

$$kk = \frac{-i}{j} \cdot \frac{j}{i} = \frac{-i}{i} = -1.$$

Both are correct *in this case*, and the matter might be passed with the single remark that the quaternions are complanar and hence commutative. This is a case, strictly speaking, of the multiplication of quaternions, but it should be observed that no rules have thus far been given for this operation; *that* was reserved for a future article. The expression was not designed to teach multiplication, but was simply an expedient for accomplis'ng a certain end, and it only remains to be shown that the manner in which it was used is admissible.

While the non-commutative principle is a characteristic of this science, yet there are exceptions to it. Thus, if α and β be two parallel vectors, we have $\alpha\beta = +\beta\alpha$, or they are commutative (ANALYST, Vol. VII, p. 125); but in no other case (p. 124). Now, considering division in the sense given by the critic (p. 187), where we have $k = i \div j$, $k = -j \div i$, it is evident that in the expression kk it is immaterial which k is written first in order. The same result becomes apparent in regard to the second member of the equation, for by beginning with the second quadrant, instead of the first we have

$$kk = \frac{-i}{j} \cdot \frac{-j}{-i} = \frac{-i}{i} = -1,$$

where the order of the factors in the second member have become practic'ly reversed, and stand in the proper order for cancellation in the *general case* of multiplication. But these are right quaternions, and although they include all that was involved in the case under consideration, yet we will show that the principle is true for all complanar quaternions. We have for unit vectors (ANALYST, Vol. VII, p. 124),

$$q = \frac{\beta}{\alpha} = \cos\theta + i\sin\theta,$$

$$q' = \frac{\delta}{\gamma} = \cos\varphi + i\sin\theta;$$

where the axis i of these quaternions will be common, since they are complanar. An inspection of the second members of these equations shows that the first multiplied by the second will give the same result as the second multiplied by the first, and we have

$$qq' = q'q = \cos(\theta+\varphi) + i\sin(\theta+\varphi);$$

hence *they are commutative.* It is unnecessary, for present purposes, to consider the general case of the multiplication of diplanar quaternions.

We observe that $\beta \div \alpha$ is not, by *definition*, equal to $\beta\alpha^{-1}$; but that it is a *result* flowing from previous definitions. That it might have been so *defined* we admit, but for the same reason it might have been defined to equal $\alpha^{-1}\beta$, a form which would have changed in an important particular the existing relation between multiplication and division. Hamilton nor Kelland so treat it. (*Lectures*, p. 124; Kelland and Tait's *Introduction to Quaternions*, p. 45. See also ANALYST, Vol. VII, p. 124.)

We have now reached the point where I am charged with giving credit to Hamilton which does not belong to him.

We quote the following from the criticism (see p. 187): "Prof. W. says, 'In division, the versor operating on the divisor line is conceived to turn it, positively about the axis of the versor $(+i)$ through an angle equal to that of the versor (θ) to coincide in direction with the dividend line. In multiplication the versor operating on the multiplier line [as α, eq. (27)] is conceived to turn it in a positive direction about the axis of the versor $(+i)$ through an angle equal to the *supplement* of the angle of the versor $[\pi-(\pi-\theta)=\theta]$, making it coincide in direction with the multiplicand line (as β).'" Prof. W. cites p. 85 of the Lectures for the above, but I can assure the readers of the ANALYST that Hamilton does not deserve the credit of it. In the 'division'

$$\frac{\beta}{\alpha} = \cos\theta + i\sin\theta,$$

the versor $\cos\theta + i\sin\theta$, so far from 'operating upon the divisor line' (α), in reality *does not operate upon any line whatever, does not act at all,* but simply *constitutes the conception* $\beta \div \alpha$. So soon as the equivalency of the vector and the right part of the quaternion is established we may change the 'quotient' $\frac{\beta}{\alpha}$ into the 'product' $\beta . \frac{1}{\alpha} = -\beta\alpha$, and *then* we have a *bona fide* 'operation', viz., β and the inversor (—) operate on α;" and on p. 188, "To recapitulate results: It is seen that in the *division* $\beta \div \alpha$ the versor $\cos\theta + i\sin\theta$, does not operate at all; in the equivalent *multiplication* $\beta\,.\,\frac{1}{\alpha}$, β and the inversor (—) operate upon α to produce the versor $\cos\theta + i\sin\theta$. Let this be compared with the first part of prof. W.'s statement.

In the multiplication $\alpha\beta$, the versor $-\cos\theta + i\sin\theta$, so far from operating upon the multiplier line α, is the result of α operating upon β — the multiplier line upon the multiplicand line, as it should be.

The assertion of the critic that "in the division $\beta \div \alpha$, the versor *does not operate upon any line*" is general, applying to right as well as oblique vectors; and strikes directly — as it was doubtless intended to do — at the substance of what we have said upon this operation. We will see what the author of the system teaches. Of right quaternions, Hamilton in his *Lectures*, pp. 61, 62, says "In regard to the right quaternion $\beta \div \alpha$, if i (that author uses j in the same sense, while we retain i to correspond with our own notation) be joined (mechanically) perpendicularly to α, and one takes hold of the axis (i) and turns α positively through a quadrant to coincide with β, the required act of version will be performed. And since the (mechanical) *agent* in producing this (mechanical) rotation, has been an axis, I now propose to denote the *versor* itself, or the CONCEIVED AGENT of the *conceived* version, or the purely geometrical rotation from α to β by i . . . . . the line turned being a subject of operation, and the turning line being a vesor or operator". This is not a literal extract, some phrases not essential to the meaning being omitted. If space permitted, we would be pleased to make extended extracts from the context; but if we read Hamilton correctly he asserts that the versor (i) operates upon α, the divisor line, turning it mechanically through a quadrant.

Next, in regard to the versor in general, we have (*Lectures*, p. 83) "The symbol i^t denotes a versor which would cause any right line in a plane perpendicular to it, to revolve in that plane through t quadrants . . . . in producing which rotation the *versor* is *conceived* to be the agent". If there were any doubt whether this *versor* or agent were an *operator*, it would be removed by the preceding extract. On page 128 of *Lectures* we have $\beta \div \alpha = i^t$, where "rotation round the base-line i from the divisor-line α to the dividend-line β, is positive". We understand that these two extracts do imply (or *express*) that i^t as a versor operates on α turning it mechanically to coincide with β. This conception is fundamental and will not be called in question. (*Elements*, p. 364, Tait's *Quaternions*, p. 38, &c.)

The critic says “$\cos\theta + i\sin\theta$ simply *constitutes the conception* $\beta \div \alpha$.” Grant it; the *conception* is one of rotation through an angle θ about the axis i. Kelland says “$\cos\theta + i\sin\theta$ is an operator of the same character as i; with this difference only, that whereas i as an operator would turn α through a right angle, $\cos\theta + i\sin\theta$ turns it in the same direction only through the angle θ (Kelland and Taits *Quat.*, p. 45). The fact that $\beta\div\alpha$ constitutes a vector-division does not prevent its being a versor, nor a quaternion; nor does the fact that i^t expresses a vector quotient prevent it from being a versor or a quaternion (Lectures, pp. 85, 124, 127, 128).

The critic, apparently willing to provide a way of escape, remarks, p. 188, that “It is possible that Prof. W. had in mind that

$$\frac{\beta}{\alpha} \cdot \alpha = (\cos\theta + i\sin\theta)\alpha = \beta.\text{”}$$

But this is not the case we had in mind, and it differs essentially from the one criticised. Here the *multiplying factor* is a versor, and the operation is among the first described by Hamilton. It is essentially the same as $\beta\div\alpha = i^t$. Hamilton says, in regard to the two forms $q\alpha = \beta$ and $q = \beta \div \alpha$, “they are indeed, *intrinsically*, the same, but present themselves under different forms (*Lectures*, pp. 90, 91).

Again, the critic says, p. 188, that in the multiplication $\alpha\beta$, α operates upon β producing $-\cos\theta + i\sin\theta$. If this be true α must operate either on the length or direction of β (*Lectures*, p. 75). It does not operate on the length, for its length is unity; neither does it operate on the direction, for it would produce a quadrantal rotation, whereas the actual rotation is θ. “A unit vector as a factor may be considered as a quadrantal versor whose plane is perpendicular to the versor.” (*Lectures*, p. 76, Tait’s *Quat.* p. 37.) Also if $\theta = 0$, we find $\alpha\alpha = -1$, which would be explained according to the critic’s principle, α operating on itself produces the symbol of reversion; a questionable explanation at least. But according to the paragraph under consideration, we would say that the inversor *minus*, operating upon the *multiplier* α turns it through the *supplement* of the angle of inversion (i. e. $180^\circ - 180^\circ$) 0° to coincide with the multiplicand α, which *result* is correct. We observe in passing that α^2 is itself an operator on a line to which it is perpendicular (*Elements*, p. 316). We conclude, therefore, almost in the words of the critic, that in the multiplication $\alpha\beta$, α does not operate upon any line whatever, *does not act at all*, but is simply the first of two factors (so called) in the order of arrangement, for forming a vector product.

Passing over some minor points in the criticism, we proceed to the main point. We observe at the outset that this is a case of the *multiplication of* *two lines.* Hamilton first discussed the case of a line multiplied by a versor producing another line, as $i^t\alpha = \beta$; after which he discussed the case $\alpha\beta = i^{2-t}$. The latter is referred to on p. 76 of *Lectures* as follows: "much less have we shown how to multiply generally *any one vector by another*". This case is first taken-up on p. 83, *after* the two cases $\beta \div \alpha = i^t$ and $i^t\alpha = \beta$ had been discussed, and the first inference is "The product $\alpha\beta$ as unit vectors is therefore a versor" (*Lectures,* p. 85). Note the difference between this case of multiplication and the preceding; here the multiplier is a line, there a versor; here the product is a versor, there a line. The next inference is "the angle of the versor is the supplement of the angle between the lines". He then sums up the matter as follows:—confining our extract to unit vectors—"The product $\kappa\lambda$ of any two unit vectors κ and λ is a versor, which versor is the power i^{2-t} of the vector-unit i, this unit i having the direction of the axis of right-hand rotation *from* the multiplier-line κ to the multiplicand-line λ; and the *supplement* t, of the exponent $2-t$ of the constant number 2, expressing the ratio of the angle of the last rotation to a right-angle (*Lectures,* p. 85). This is the paragraph to which we referred in support of our conclusion, but the critic assures us that Hamilton should not have the credit of sustaining us. The *statical relations* (so to speak) of all the elements are evidently the same in Hamilton's statement as in our conclusion; and hence the obj. must be to our *dynamic* statem't "The versor operating on the multiplier-line". But Hamilton, only two pages preceding, describes a versor as an agent—an operator—and here describes the versor and the amount of rotation to be performed. The reader can judge if too much credit has been given to Hamilton.

But, as if to show still further my error, the critic says, p, 188, "We do *not* have $\alpha\beta.\alpha = \beta$". This is merely setting up a man of straw and demolishing *him,* for we have not asserted that it was true. But we do assert that if $\alpha\beta$ as a versor turns α through an angle equal to $\pi - \angle\alpha\beta$, it will produce β. We did *not* refer to α being operated upon as a multiplicand-line, but if it is desired to place it in that position, let it be done legitimately; thus

$$i^{2-t} = \alpha\beta = -\alpha\beta^{-1} = -\frac{\alpha}{\beta};$$

and taking the reciprocal, we have

$$\frac{\beta}{\alpha} = -(i)^{t-2} = -(-i)^{2-t} = i^t \text{ [ANALYST, Vol. VII. p. 122, eq. (20);]}$$

$$\therefore\ \beta = i^t\alpha;$$

a form admitted by Mr. Christie as expressing a legitimate operation, and this we claim is a demonstration that if it can be asserted, in any sense, that the versor i^t as a multiplier operating upon α as a multiplicand produces β;

then it can with equal force and in the same sense be asserted that the versor i^{2-t} operating upon the multiplier-line α, will by turning it through the angle t (the supplement of $2-t$) produce the multiplicand line; for one follows the other by legitimate analysis. We do not therefore abandon our position, but affirm it; still, we will relieve Hamilton and all others of any responsibility for our conclusion by changing it to the following: —In the multiplication $\alpha\beta = i^{t-2}$, *we* conceive that the axis i is attached (mechanically) perpendicularly to the multiplier-line (α), and that the versor (i^{2-t}), turns the latter (mechanically) positively through an angle equal to the supplement of the exponent $2 - t$, that is, through the angle t, making it coincide with the multiplicand-line (β).

In regard to the matter referred to by Prof. Johnson in the last March No. of the Analyst, I should have said that a quaternion may be expressed in terms of the four elements *tensor*, *scalar*, *versor*, and *vector*, as was finally shown in my equation (54), p, 127 of the preceding Vol. of the Analyst, which equation is

$$q = Tq(SUq + VUq).$$

Response to a Request, by the Editor.—We are requested by a subscriber to furnish a solution to the following problem:—

"A conical hay-stack, altitude a and radius of base r, is to be divided horizontally into three equal parts, by weight; if the density of horizontal strata is everywhere proportional to their distance from the vertex, what must be the vertical height of each part?"

Put x = the distance of any horizontal section from the vertex, then will its area be $\pi(rx \div a)^2$; and if we assume the density at the vertex to be unity and the thickness of the horizontal strata to be dx, we shall have for an element of the cone at the distance x from its vertex, $\pi(rx \div a)^2 x dx$; therefore, if x be the height of the upper part, we must have

$$\int_0^x \left(\frac{rx}{a}\right)^2 \pi x dx = \tfrac{1}{3}\int_0^a \left(\frac{rx}{a}\right)^2 \pi x dx;$$

whence we find

$$x = a(\tfrac{1}{3})^{\frac{1}{4}}.$$

And if x = the height of the upper two thirds of the stack we shall find

$$x = a(\tfrac{2}{3})^{\frac{1}{4}}.$$

Hence, the heights of the upper, middle and lower parts must be, respectively, $a(\frac{1}{3})^{\frac{1}{4}}$, $a[(\frac{2}{3})^{\frac{1}{4}}-(\frac{1}{3})^{\frac{1}{4}}]$, and $a[1-(\frac{2}{3})^{\frac{1}{4}}]$.

Query.—If in the foregoing prob. the density of the strata is supposed to vary as the superincumbent weight, how must the division be made?